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AN APPLICATION OF λ-METHOD ON SHAFER-FINK'S INEQUALITY

Branko J. Malešević

In the paper λ-method Mitrinović-Vasić is applied aiming to improve Fink's inequality, and Shafer's inequality for arcus sinus function is observed.

In monography [**1**, p. 247] SHAFER's inequality is stated:

$$\frac{3x}{2+\sqrt{1-x^2}} \leq \operatorname{asin} x \qquad (0 \leq x \leq 1). \tag{1}$$

The equality holds only for $x = 0$. In paper [**2**] FINK has proved the inequality:

$$\operatorname{asin} x \leq \frac{\pi x}{2+\sqrt{1-x^2}} \qquad (0 \leq x \leq 1). \tag{2}$$

The equality holds at both ends of the interval $x = 0$ and $x = 1$. Let us notice that from the inequality (1) and (2) the function $g(x) = \operatorname{asin} x$ is bounded by the corresponding functions from the two-parameters family of functions:

$$\Phi_{a,b}(x) = \frac{ax}{b+\sqrt{1-x^2}} \qquad (0 \leq x \leq 1), \tag{3}$$

for some values of parameters $a, b > 0$. For the values of parameters $a, b > 0$ the family $\Phi_{a,b}(x)$ is the family of raising convex functions on variable x on interval $(0, 1)$. Let us apply λ-method MITRINOVIĆ-VASIĆ [**1**] on considered two-parameters family $\Phi_{a,b}$ in order to determine the bound of function $g(x)$ under the following conditions:

$$\Phi_{a,b}(0) = g(0) \quad \text{and} \quad \frac{\mathrm{d}}{\mathrm{d}x}\,\Phi_{a,b}(0) = \frac{\mathrm{d}}{\mathrm{d}x}\,g(0). \tag{4}$$

It follows that $a = b + 1$. In that way we get one-parameter subfamily:

$$f_b(x) = \Phi_{b+1,b}(x) = \frac{(b+1)x}{b+\sqrt{1-x^2}} \qquad (0 \leq x \leq 1), \tag{5}$$

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according to parameter $b > 0$, which fulfills condition (4). For family (5) the following equivalence is true:

$$f_{b_1}(x) < f_{b_2}(x) \;\Leftrightarrow\; b_1 > b_2. \tag{6}$$

Let us consider one-parameter function of the distance:

$$h_b(x) = f_b(x) - g(x) = \frac{(b+1)x}{b+\sqrt{1-x^2}} - \operatorname{asin} x \qquad (0 \le x \le 1), \tag{7}$$

as well as its derived function:

$$\frac{\mathrm{d}}{\mathrm{d}x}\, h_b(x) = \frac{\big(\sqrt{1-x^2} - (b^2-b-1)\big)\cdot x^2}{(b+\sqrt{1-x^2}\,)^2(1-x^2+\sqrt{1-x^2}\,)} \qquad (0 \le x < 1).$$

Then, it holds that $h_b(0) = 0$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}h_b(0) = 0$. In further consideration let us use the equivalence:

$$\frac{\mathrm{d}}{\mathrm{d}x}h_b(x) \ge 0 \Leftrightarrow \sqrt{1-x^2} \ge (b^2-b-1). \tag{8}$$

The least upper bound of the function $g(x)$ from family (5), on the basis of equivalence (6), we get for the maximum value of parameter b for which $g(x) < f_b(x)$ is true. Let us notice $h_b(1) = 1 + \dfrac{1}{b} - \dfrac{\pi}{2} \ge 0$ iff $b \in \Big(0, \dfrac{2}{\pi-2}\Big]$. If $b \in \Big(0, \dfrac{1+\sqrt{5}}{2}\Big]$ the right side of equivalence (8) is always true. If $b \in \Big(\dfrac{1+\sqrt{5}}{2}, \dfrac{2}{\pi-2}\Big]$ the right side of equivalence (8) is true for $x \in (0, d]$ where $d = d(b) = \sqrt{-b^4+2b^3+b^2-2b}$. For the maximum value of parameter $b_1 = \dfrac{2}{\pi-2}$ we find that $d_1 = d(b_1) \cong 0.948 \in (0,1)$. The function $h_{b_1}(x)$ fulfills $h_{b_1}(0) = h_{b_1}(1) = 0$ and reaches the maximum for $x = d_1$. Therefore for the value of the parameter $b_1 = \dfrac{2}{\pi-2}$ the function $f_{b_1}(x)$ is the least upper bound of the function $g(x)$ from the family (5). Thus, inequality is proved:

$$\operatorname{asin} x \le \frac{\frac{\pi}{\pi-2}x}{\frac{2}{\pi-2} + \sqrt{1-x^2}} \qquad (0 \le x \le 1). \tag{9}$$

The equality holds at the both ends of the interval $x = 0$ and $x = 1$. The maximum distance of the function $f_{b_1}(x)$ from the function $g(x)$ is reached for $x = d_1$ and it equals $h_{b_1}(d_1) \cong 0.013$. It is directly verified that $f_{b_1}(x) = \Phi_{b_1+1,b_1}(x) < \Phi_{\pi,2}(x)$. Thus, the given upper bound is better than the one shown in the paper [**2**].

The greatest lower bound of the function $g(x)$ from the family (5), on the basis of equivalence (6), we get for the minimum value of parameter b for which it holds $f_b(x) < g(x)$. If $b \in (b_1, 2)$ then the function $h_b(x)$ has a root on $(0,1)$. If $b \geq 2$ on the basis of equivalence (8) we can conclude: $\dfrac{\mathrm{d}}{\mathrm{d}x} h_b(x) \leq 0$. Thus, for the value of parameter $b_2 = 2$ SHAFER's function $f_{b_2}(x)$ is the greatest lower bound of the function $g(x)$ from the family (5).

REFERENCES:

1. D. S. MITRINOVIĆ, P. M. VASIĆ: *Analytic inequalities.* Springer–Verlag 1970.
2. A. M. FINK: *Two inequalities.* Publikacije ETF Ser. Mat. **6** (1995), 48–49.

Faculty of Electrical Engineering,
University of Belgrade,
P.O.Box 816, 11001 Belgrade,
Yugoslavia

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